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EQUIVALENT CONSTRAINTS FOR DISCRETE SETS.(U)
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EQUIVALENT CONSTRAINTS FOR DISCRETE SETS

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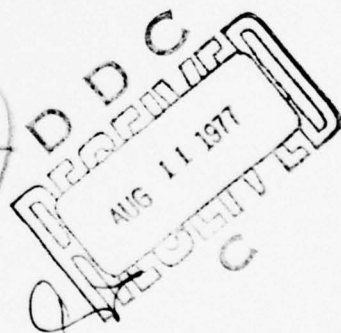
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ABSTRACT

Two types of "simplifications" are considered for constraints over discrete sets: (1) replacing real data by "equivalent" rational data, and (2) collapsing a system of linear or nonlinear equations into an "equivalent" single equation. Such transformations are not only of computational interest, but also provide some interesting insights into stability properties of integer programs.

EQUIVALENT CONSTRAINTS FOR DISCRETE SETS

R. R. Meyer

1. Introduction

Gould and Rubin [6] developed a procedure for perturbing the (real) data of a system of linear constraints on a finite set of rational n-vectors so as to obtain an equivalent set of constraints with rational data (see Theorem 1 below). The purpose of Section 2 of this paper is to describe how a related result of Meyer and Wage [11] for the unbounded case may be used in an alternative and somewhat simpler derivation of the results of Gould and Rubin. The approach used in Section 2 is then generalized in Sections 3-5 to extend certain ideas of Bradley [1] and others for collapsing systems of equations into "equivalent" single equations.

2. "Rationalizing" Linear Equations

The following result is the main theorem of Gould and Rubin [6]:

Theorem 2.1: Let

$$F_R \equiv \{x | Ax \le b, Dx = e, x \in X^*\},$$

where the matrices and vectors A, b, D, and e are comprised of reals, and X^* is a non-empty, <u>finite</u> subset of Q^n, the set of rational n-vectors. Then there exist rational matrices and vectors $\hat{A}$, $\hat{b}$, $\hat{D}$, $\hat{e}$ (which may be chosen arbitrarily close to A, b, D, e respectively) such that

$$F_R = F_Q \equiv \{x | \hat{A}x \le \hat{b}, \hat{D}x = \hat{e}, x \in X^*\}$$

To establish this result, the inequality constraints may be dealt with by perturbing the data in a fairly straightforward manner, but the equations require

rather more delicate considerations. To deal with the equations, Gould and Rubin established the following Lemma:

<u>Lemma 2.2:</u> Let $r \in \mathbb{R}^n$, $\hat{r}_0 \in Q^1$, and define $Q(r) \equiv \{x | \; x \in Q^n, r x = \hat{r}_0\}$. Then there exist rational vectors r which may be chosen arbitrarily close to r such that $Q(r) \subset \{x | \; x \in Q^n, \; \hat{r} x = \hat{r}_0\} \equiv Q(\hat{r})$.

Given this Lemma, by selecting an $\hat{r}$ close enough to r so that no element of $X^* \backslash Q(r)$ can satisfy $\hat{r} x = \hat{r}_0$, the relation $Q(r) \cap X^* = Q(\hat{r}) \cap X^*$ is established, and the main theorem may be proved by considering the equations and inequalities one at a time.

Gould and Rubin proved Lemma 2.2 by extracting a maximal set of linearly independent vectors of $Q(r)$ and considering certain algebraic transformations involving the inverse of a submatrix of that maximal set of columns. The procedure to be described below does not require any matrix operations and works directly with the data $(r, \hat{r}_0)$ rather than the vectors of $Q(r)$. The method essentially consists of re-writing the equation $r x = \hat{r}_0$ in such a way that Lemma 2.2 may be proved by selecting rationals sufficiently close to the irrationals in the <u>re-formulated</u> version of $r x = \hat{r}_0$. (Gould and Rubin note that the result of Lemma 2.2 cannot be obtained by simply choosing $\hat{r}$ close to r, so the nature of the re-formulation of the equation is critical to the proof.) Lemma 2.2 is trivially true if $Q(r) = \phi$, so we shall assume $r \neq 0$ and $Q(r) \neq \phi$.

The "critical" constants are identified by determining a minimal cardinality index set I such that <u>each</u> r_j may be expressed as a <u>rational</u> combination of r_i with $i \in I$ (the r_i with $i \in I$ may be thought of as a <u>basis</u> over the rational field for the elements of r). Such an index set I corresponds

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to the <u>rationally independent</u> elements (i. e. , no non-trivial rational combination. is 0) of r, and may be derived by the usual procedure for extracting a linearly inde-pendent set. Once this is done, the equation $rx = \hat{r}_0$ is written as

$$\sum_{j=1}^{n} (\sum_{i \in I} \alpha_{i,j} r_i) x_j = \hat{r}_0 ,$$

where the $\alpha_{i,j}$ are rational, and then as $\sum_{i \in I} r_i (\sum_{j=1}^{n} \alpha_{i,j} x_j) = \hat{r}_0$. Since we are assuming that $Q(r) \neq \phi$, there exists rationals $\bar{x}_j$ such that

$$\hat{r}_0 = \sum_{i \in I} r_i (\sum_{j=1}^{n} \alpha_{i,j} \bar{x}_j) ,$$

and, for notational convenience, we define the rationals

$$\beta_i = \sum_{j=1}^{n} \alpha_{i,j} \bar{x}_j \qquad (i \in I) .$$

(Note that this approach does not require that $\hat{r}_0$ be rational.) Then, as shown in [11], each element of $Q(r)$ must satisfy the system of equations

$$(2.1) \qquad \sum_{j=1}^{n} \alpha_{i,j} x_j = \beta_i \qquad (i \in I),$$

(note that every rational solution of this <u>system</u> also is in $Q(r)$, so that the system (2.1) provides a rationalization of $Q(r)$ that is valid in the <u>unbounded</u> case) so that for <u>arbitrary</u> constants r'_i ,

$$(2.2) \qquad Q(r) \subset \{ x | x \in Q^n, \sum_{i \in I} r'_i (\sum_{j=1}^{n} \alpha_{i,j} x_j) = \sum_{i \in I} r'_i \beta_i \} .$$

Since the r'_i may be chosen as rationals arbitrarily close to the r_i, the relation (2.2) yields Lemma 2.2. (Note also that if the r'_i are chosen as the elements of any <u>rationally independent</u> set, then the subset relation in (2.2) may be replaced by an equation. This result will be generalized in Section 3.)

<u>Example 2.1</u> : Consider the equation

$$\sqrt{2} x_1 + (1 - \sqrt{2}) x_2 = 1 ,$$

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and re-write it as

$$\sqrt{2}\, x_1 + (1 - \sqrt{2}\,)x_2 = \sqrt{2} + (1 - \sqrt{2}\,).$$

From the latter formulation and the rational independence of $\sqrt{2}$ and $1 - \sqrt{2}$, it follows that any rational solution of the equation must satisfy $x_1 = 1$, $x_2 = 1$, i.e., the equation has a unique solution over the rationals. Thus, for any constants $\hat{r}_1$ and $\hat{r}_2$, the vector $(1,1)^T$ will be contained in the solution set of

$$\hat{r}_1 x_1 + \hat{r}_2 x_2 = \hat{r}_1 + \hat{r}_2 .$$

It might be noted that for this example the rationalization procedure of Gould and Rubin would yield

$$(\sqrt{2} + \varepsilon)x_1 + (1 - \sqrt{2} - \varepsilon)x_2 = 1 ,$$

where ε is chosen so that the coefficients are rational. This example illustrates that the procedure described in this paper may result in a modified RHS if r is comprised entirely of irrationals, whereas the rationalization procedure in [6] will leave the RHS unchanged, but will require as a consequence that the perturbation of the coefficients be of a more restricted form. ∎

Example 2.2: $\frac{1}{3} x_1 + \frac{1}{5} x_2 + \sqrt{2}\, x_3 + \sqrt{5}\, x_4 = \frac{1}{2}$.

The rationalization procedure described above yields the equation:

$$\frac{1}{3} x_1 + \frac{1}{5}x_2 + \hat{r}_3 x_3 + \hat{r}_4 x_4 = \frac{1}{2} ,$$

where $\hat{r}_3$ and $\hat{r}_4$ are arbitrary rationals. The rationalization procedure of Gould and Rubin yields the same result, but requires somewhat more algebraic manipulation to arrive at that result. This example illustrates that, in the case in which r contains at least one non-zero rational and only rationally independent irrationals, the rationalization procedure can be accomplished by

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simply replacing the irrationals by arbitrary rationals.   ∎

Note that while the <u>system</u> of equations (2.1) is equivalent to the single equation $rx = \hat{r}_0$ over the rationals, i.e., $Q(r) = \{x \,|\, x \in Q^n, \sum\limits_{j=1}^{n} \alpha_{i,j} x_j = \beta_i \,(i \in I)\}$, in general there does not exist a <u>single</u> equation with rational coefficients whose solution set over <u>all</u> the rationals will be $Q(r)$. For example, if $\hat{r}_0 \neq 0$ and r_k is irrational for some index k, then for any rational vector $\hat{r}$ with $\hat{r}_k \neq 0$, $Q(\hat{r})$ contains a multiple of the kth unit vector, but $Q(r)$ does not. On the other hand, the approach used above indicates how the system (2.1) may be "collapsed" into a single equation having the same intersection with some <u>finite</u> set. In the next section we will indicate how the constraint combination approach can be generalized in a number of interesting ways.

3. "Collapsing" Systems of Equations into Single Equations

In this section we will consider conditions on functions f_1 and f_2 <u>real-valued</u> on a set X such that the set defined by

$$E \equiv \{x \,|\, x \in X, \; f_1(x) = 0, \; f_2(x) = 0\}$$

coincides with the set

$$E(\varepsilon) \equiv \{x \,|\, x \in X, \; f_1(x) + (1 + \varepsilon)f_2(x) = 0\}$$

for certain values of the parameter ε. Using this approach, a <u>system</u> of m equations may be "collapsed" into a single equation by combining two equations at a time. While the technique used in Section 2 took advantage of the linearity of the functions involved and the finiteness of the set X^*, these restrictions can be relaxed in various ways, and the results to be obtained also generalize some of the results of Bradley [1]. The approach also differs

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from Bradley's, in that it is based on the perturbation ideas of the previous section rather than the number theoretic properties used in [1].

In order to identify values for ε such that $E = E(\varepsilon)$, it is convenient to define a "prohibited" set

$$P \equiv \{v \mid v = \frac{f_1(x) + f_2(x)}{-f_2(x)} \text{ for some } x \in X'\},$$

where $X' \equiv \{x \mid x \in X, f_2(x) \neq 0\}$. (Note that P is determined by f_1, f_2, and X.)

The motivation for this definition is the following Lemma:

Lemma 3.1: $E = E(\varepsilon)$ if and only if $\varepsilon \notin P$.

Proof: Clearly $E \subseteq E(\varepsilon)$ for all ε, so we need only show $E(\varepsilon) \subseteq E$. If $\varepsilon \notin P$ and $\bar{x} \in E(\varepsilon)$, then $f_2(\bar{x}) = 0$ implies $f_1(\bar{x}) = 0$, so suppose $f_2(\bar{x}) \neq 0$. Then $\varepsilon = (f_1(\bar{x}) + f_2(\bar{x}))/(-f_2(\bar{x}))$ and $\bar{x} \in X'$ contradict $\varepsilon \notin P$. Conversely, if $\varepsilon \in P$, there exists an $\bar{x} \in X'$ such that $\varepsilon = (f_1(\bar{x}) + f_2(\bar{x}))/(-f_2(\bar{x}))$, so that $\bar{x} \in E(\varepsilon)$ but $\bar{x} \notin E$.   ∎

The Lemma implies that a valid combination of the constraints exists if $P \neq R^1$, so the results below are based on specifying conditions on f_1, f_2, and X that yield identifiable "gaps" in P. Qualitatively, three types of results will be considered: (1) $P \subseteq Q^1$, (2) P is finite, and (3) P contains a gap near 0.

Our first result deals with the case in which $P \subseteq Q^1$ and generalizes the results of Section 2.

Theorem 3.2   If, for $i = 1$ and 2, $f_i(x)$ is <u>rational</u> for $x \in X$, then $E = E(\varepsilon)$ for all irrational ε. More generally, if $f_1, \ldots, f_m$ are rational-valued on X and $\varepsilon_1, \ldots, \varepsilon_m$ are rationally independent, then

$$\{x \,|\, x \in X, \ f_1(x) = 0, \ \ldots, \ f_m(x) = 0\} \ =$$

$$\{x \,|\, x \in X, \ \varepsilon_1 f_1(x) + \ldots + \varepsilon_m f_m(x) = 0\} \ .$$

Proof: Follows directly from the definition of rational independence. ∎

Since the ε_i may be chosen arbitrarily close to 1 , note that Theorem 3.2 says that there exist <u>equivalent</u> formulations whose single equation is arbitrarily "close" to the sum of the original equations. Note, however, that simply summing the equations will, in general, <u>not</u> lead to an equivalent formulation.

<u>Finiteness</u> of P can be guaranteed by assuming that X is a finite set or that f_1 and f_2 take on only finitely many different values over X. When P is finite, Lemma 3.1 shows that "weighting" and adding two constraints leaves the feasible set unchanged except for a <u>finite</u> number of "prohibited" values of ε . While P itself may be difficult to compute, gaps in P may be easier to identify.

<u>Theorem 3.3:</u> For $i = 1, 2$, let $f_i : X \to Y_i$, where Y_i is a <u>finite</u> subset of $\mathbb{R}^1$ for each i . Then there exist positive constants $\underline{M}$ and $\bar{M}$ such that if $|\varepsilon| \in (0, \underline{M})$ or $|\varepsilon| > \bar{M}$, then $E = E(\varepsilon)$.

<u>Proof:</u> The result follows immediately from the finiteness of P . ∎

In particular, Theorem 3.3 will hold if X is any <u>finite</u> set. Ruling out the trivial cases in which $f_1(x) + f_2(x) = 0$ for all $x \in X$ (in this case any $\varepsilon \neq 0$ may be used, and in particular $E = E(-1) = \{x \,|\, x \in X, \ f_1(x) = 0\} = \{x \,|\, x \in X, \ f_2(x) = 0\}$) or $f_2(x) = 0$ for all $x \in X$ (in this case any ε may be used, and in particular $E = E(-1) = \{x \,|\, x \in X, \ f_1(x) = 0\}$), expressions for $\underline{M}$ and $\bar{M}$ may be derived, since it is clear from the definition of P that in

Theorem 3.3 we may take $\underline{M} = \gamma\omega^{-1}$, where

$$\gamma \equiv \inf |f_1(x) + f_2(x)|$$
$$\text{s.t. } f_1(x) + f_2(x) \neq 0, \quad x \in X,$$
$$\omega \equiv \sup |f_2(x)|$$
$$\text{s.t. } x \in X,$$

and $\bar{M} = \alpha\beta^{-1}$, where

$$\alpha \equiv \sup |f_1(x) + f_2(x)|$$
$$\text{s.t. } x \in X,$$
$$\beta \equiv \inf |f_2(x)|$$
$$\text{s.t. } f_2(x) \neq 0, \quad x \in X.$$

Sharper results can, of course, be obtained by considering appropriate sign restrictions on $f_1 + f_2$ and f_2, and, on the other hand, cruder estimates of $\underline{M}$ and $\bar{M}$ can be obtained by using estimates for $\gamma, \omega, \alpha,$ or β derived from <u>relaxed</u> versions of the corresponding optimization problems. (For example, if X is comprised of the vertices of a cube, X may be replaced in the defining optimization problem by the cube itself.)

Even if the finiteness hypothesis of Theorem 3.3 does <u>not</u> hold, the relation $E = E(\varepsilon)$ will hold for $|\varepsilon|$ in the ranges described above, provided that $\gamma, \omega, \alpha,$ and β are positive and finite.

<u>Corollary 3.4:</u> If γ and ω are positive and finite, then $E = E(\varepsilon)$ provided that $|\varepsilon| \in (0, \gamma\omega^{-1})$. If α and β are positive and finite, then $E = E(\varepsilon)$ provided that $|\varepsilon| > \alpha\beta^{-1}$.

<u>Proof:</u> It is easily verified that if ε satisfies the conditions of the Corollary, then $\varepsilon \notin P$. ∎

Positivity of γ and β can be guaranteed by assuming that f_1 and f_2 are <u>integer-valued</u> over X, but it is also sufficient in this regard to assume that the f_i are finite sums of the form $\sum r_j h_j(x)$ where the r_j are rational and the h_j are integer-valued over X. If, in addition, certain <u>boundedness</u> properties are assumed for the f_i, it will follow that $E = E(\varepsilon)$ for $|\varepsilon|$ positive and sufficiently small. In particular, it suffices to have f_2 bounded <u>both from above and below</u> on X, or <u>each</u> f_i bounded <u>either</u> from above or below on X. In the former case we have:

Corollary 3.5: If $\gamma > 0$ and if f_2 is bounded both from above and below on X, then $E = E(\varepsilon)$ for all ε such that $|\varepsilon| \in (0, \gamma\omega^{-1})$.

To deal with the case of <u>one-sided</u> bounds on the f_i, we will consider only the case in which both f_i are bounded from below, since the other cases can be handled by replacing the constraint $f_i(x) = 0$ by $-f_i(x) = 0$ as needed.

Theorem 3.6: If $\gamma > 0$ and if there exist non-negative constants ℓ_1 and ℓ_2 such that, for $i = 1$ and 2, $f_i(x) \geq -\ell_i$ for all $x \in X$, then $E = E(\varepsilon)$ for all sufficiently small $\varepsilon > 0$.

Proof: Define

$$\omega_2 \equiv \sup f_2(x)$$
$$\text{s.t.} \quad f_1(x) < 0, \quad f_1(x) + f_2(x) < 0, \qquad x \in X,$$

and note that the constraint $f_1(x) + f_2(x) < 0$ implies $f_2(x) < \ell_1$, so that $\omega_2 \leq \ell_1$. It will be shown that it suffices to choose $\varepsilon > 0$ such that $\max\{\varepsilon\omega_2, \varepsilon\ell_2\} < \gamma$. (Note that $\omega_2 \leq \ell_1$ implies that ε may be chosen so that $\varepsilon \cdot \max\{\ell_1, \ell_2\} < \gamma$.) Suppose that ε is so chosen and that $\varepsilon \in P$. Let $\bar{x} \in X'$ be chosen so that

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$$f_1(\bar{x}) + f_2(\bar{x}))/(-f_2(\bar{x})) = \varepsilon$$

and thus

(3.1) $$f_1(\bar{x}) + f_2(\bar{x}) = -\varepsilon\, f_2(\bar{x}).$$

If $f_2(\bar{x}) > 0$, then $f_1(\bar{x}) < 0$, $f_1(\bar{x}) + f_2(\bar{x}) < 0$, and the absolute value of the RHS of (3.1) lies in the interval $(0,\ \varepsilon\,\omega_2]$, whereas the absolute value of the LHS is at least γ, which leads to a contradiction since $\varepsilon\,\omega_2 < \gamma$. If $f_2(\bar{x}) < 0$, we obtain a contradiction since $\varepsilon\,\ell_2 < \gamma$. ∎

Corollary 3.7: If, for $i = 1,2$, $g_i(x)$ is non-negative and <u>integer-valued</u> for all $x \in X$, and r_1 and r_2 are non-negative integers satisfying $0 \le r_2 \le r_1$, then

$$G \equiv \{x \,|\, x \in X,\ g_1(x) = r_1,\ g_2(x) = r_2\} \ =$$

$$\{x \,|\, x \in X,\ g_1(x) + [1 + (r_1 + 1)^{-1}]\, g_2(x) \ = \ \frac{r_1^2 + r_1 r_2 + r_1 + 2r_2}{r_1 + 1}\}.$$

If, in addition, $0 \le r_2 < r_1$, then

$$G = \{x \,|\, x \in X,\ g_1(x) + (1 + r_1^{-1})\, g_2(x) = \frac{r_1^2 + r_1 r_2 + r_2}{r_1}\}\ .$$

Proof: For the first conclusion, apply Theorem 3.6 with $f_i(x) = g_i(x) - r_i$, noting that $\gamma \ge 1$. For the second conclusion note that integrality implies $\omega_2 \le r_1 - 1$. ∎

Note that by Corollary 3.7, when $0 \le r_2 < r_1$ the set

$$\{x \,|\, x \in X,\ g_1(x) = r_1,\ g_2(x) = r_2\}$$

coincides with the set

$$\{x \,|\, x \in X,\ r_1 g_1(x) + (r_1+1)g_2(x) = r_1^2 + (r_1+1)r_2\}\ .$$

Thus, an <u>integer</u> combination of the two constraints into a single equivalent constraint can be obtained by weighting the first constraint by r_1 and

the second by r_1+1. (By making use of number theoretic properties, sharper results can be obtained in some cases. These results will be developed in the next section.)

For the next result of this section, we consider the special case of linear constraints. This result yields some insight into the stability properties of integer programs. On the one hand, the existence of equivalent constraint combinations implies a degree of stability, but on the other hand, the limit of these combinations (involving the sum of the equations) will generally have a larger feasible set, which implies a degree of instability.

Corollary 3.8: Let $E^* \equiv \{x \mid Dx = e, x \in X \}$, where X is a finite subset of $\mathbb{R}^n$ and let s denote the vector obtained by summing the rows of D, and σ denote the sum of the elements of e. Then there exist $\hat{s}$ and $\hat{\sigma}$ (which may be chosen arbitrarily close to s and σ respectively) such that $E^* = \{x \mid \hat{s} x = \hat{\sigma}, x \in X \}$. If, in addition, $X \subseteq Q^n$, then, in addition to the other criteria, one may also require that $\hat{s} \in Q^n, \hat{\sigma} \in Q^1$.

Proof: The results are an immediate consequence of Theorem 2.1 and Theorem 3.3. ∎

It is easily seen that inequality constraints over finite sets can be collapsed after converting them into equations by adding slacks in the usual way, since the slacks will also have values in a finite set. Note, however, that a straightforward conversion of the resulting single equation (containing slack variables) into an equivalent single inequality in the original variables is, in general, not possible since, regardless of the values of ε, the set

$$\{x \mid f_1(x) \leq 0 , f_2(x) \leq 0, x \in X\}$$

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will in general be a proper subset of

$$\{x \mid f_1(x) + (1+\varepsilon) f_2(x) \leq 0 , x \in X\}$$

even when X is finite.

By mimicing an approach used by Bradley [1] for integer-valued functions, Theorem 3.3 can be used to establish a result similar to a Lagrange-multiplier theorem.

<u>Corollary 3.9</u>: Let f_i (i = 0,...,m) be real-valued functions defined on a set X with the property that, for each i, there exists a <u>finite</u> set Y_i such that $x \in X$ implies $f_i(x) \in Y_i$; then there exist multipliers $\lambda_1, \ldots, \lambda_m$ (each of which may be chosen arbitrarily close to 1) such that the problem

$$\min f_0(x)$$

$$\text{s.t.} \quad f_i(x) = 0 \quad (i = 1, \ldots, m)$$
$$x \in X$$

is equivalent to the problem

$$\min f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x)$$

$$\text{s.t.} \quad f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) \in Y_0 .$$
$$x \in X$$

<u>Proof</u>: The problem

$$\min f_0(x)$$

$$\text{s.t.} \quad f_i(x) = 0 \quad (i=1, \ldots, m)$$
$$x \in X$$

is equivalent to

$$\min_{x,z} z$$

$$\text{s.t. } -z + f_0(x) = 0$$

$$f_i(x) = 0 \qquad (i = 1,\ldots,m)$$

$$x \in X, \ z \in Y_0$$

By using Theorem 3.3, the constraints of the latter problem may be combined to yield the equivalent problem:

$$\min_{x,z} z$$

$$\text{s.t } -z + f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) = 0 \qquad (i = 1,\ldots,m)$$

$$x \in X, \ z \in Y_0 .$$

The conclusion of the Corollary is then obtained by using the substitution

$$z = f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) . \qquad \blacksquare$$

4. A Number Theoretic Approach for Integer-Valued Functions.

Previously published results in constraint combination (e.g. [1], [2], [4], [9], [12]) have generally been based on the use of number theoretic properties. In this section we will indicate how number theoretic properties may be used in conjunction with the approach of the previous section to obtain some new results for the case in which f_1 and f_2 are integer-valued.

Rather than dealing as in the preceding section with perturbations involving a single (possibly non-integer) weight $(1+\varepsilon)$, constraint combination with two integer weights will be considered. Thus, we define

$$E'(r,s) \equiv \{x \mid r \cdot f_1(x) + s \cdot f_2(x) = 0 , \ x \in X\} ,$$

and seek sufficient conditions for $E = E'(r,s)$, where, as in Section 3,

$$E \equiv \{x \mid f_1(x) = 0, \ f_2(x) = 0, \ x \in X\}.$$

Our first result is the analog of Lemma 3.1 for this case and involves a translate P' of the prohibited set defined by $P' \equiv \{v \mid v = -f_1(x)/f_2(x) \text{ for some } x \in X'\}$, where $X' \equiv \{x \mid x \in X, f_2(x) \neq 0\}$.

Lemma 4.1: If $r \neq 0$, then

$$E = E'(r,s) \text{ if and only if } s/r \notin P' .$$

Proof: Analogous to proof of Lemma 3.1. ∎

Proceeding along the lines of Section 3, we wish to develop sufficient conditions for <u>integer</u> pairs (r,s) with $r \neq 0$ to have ratios s/r <u>not</u> in P'. (It is possible to develop an analogous set of results under the hypothesis $s \neq 0$, but to avoid cluttering the development we shall consider only the case in which $r \neq 0$. Note also that if X is infinite, then P' may contain all rationals and an equivalent rational combination of constraints may not exist.) For the remainder of this section we assume that f_1 and f_2 are <u>integer-valued</u> on X. (Note, however, that the domain X need <u>not</u> consist of integer vectors.) The following Lemma establishes a key property of elements of P'. (Recall that two integers α, β are relatively prime if their greatest common divisor ($\gcd(\alpha,\beta)$) is 1, and note that, under the assumption that the f_i are integer-valued, each element v of P' may be represented in the form s/r, where r and s are relatively prime integers.)

Lemma 4.2: If r and s are <u>relatively prime</u> integers and $s/r \in P'$, then there exists an <u>integer</u> $k \neq 0$ and an $\bar{x} \in X'$ such that $f_1(\bar{x}) = k \cdot s$ and $f_2(\bar{x}) = -k \cdot r$.

Proof: Since $s/r \in P'$, there exists an $\bar{x} \in X'$ such that $-f_1(\bar{x})/f_2(\bar{x}) = s/r$, and thus there exists a constant $k \neq 0$ such that $f_1(\bar{x}) = k \cdot s$ and $f_2(\bar{x}) = -k \cdot r$.

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(Note that if $s = 0$, then $r = 1$, and the conclusion is immediate.) Since r and s are relatively prime and $f_1(\bar{x})$ and $f_2(\bar{x})$ are integer, $|k|$ is the "generalized greatest common divisor" of $f_1(\bar{x})$ and $f_2(\bar{x})$, and is thus integer (see [11]). Alternatively, one could note that $-r \cdot f_1(\bar{x}) = s \cdot f_2(\bar{x})$, and obtain the result by appropriately grouping the prime factors on both sides. ∎

Proceeding along the lines of the previous section, we develop a result based on identifying values not in P'.

Theorem 4.3: Let r and s be relatively prime integers such that one of the following sets of conditions is satisfied:

(a) $f_2(x) \in [-\ell_2, u_2]$ for all $x \in X$; $|r| > \frac{1}{m} \max \{\ell_2, u_2\}$ for an integer $m \geq 1$; and, (in the case $m \geq 2$) for $|j| = 1, \ldots, m-1$, $\bar{x} \in X$ and $f_1(\bar{x}) = -j \cdot s$ imply $f_2(\bar{x}) \neq j \cdot r$;

(b)   for $i = 1, 2$ and for all $x \in X$, there exist non-negative integers ℓ_i such that $f_i(x) \geq -\ell_i$; $r > \ell_2 / m$ and $s > \ell_1 / m$ for an integer $m \geq 1$; and (in the case $m \geq 2$) for $|j| = 1, \ldots, m-1$, $\bar{x} \in X$ and $f_1(\bar{x}) = -j \cdot s$ imply $f_2(\bar{x}) \neq j \cdot r$;

then $E = E'(r, s)$.

Proof: Assume that $E \neq E'(r, s)$. The hypotheses imply $r \neq 0$, so by Lemmas 4.1 and 4.2, there exists an $\bar{x} \in X'$ and an integer $k \neq 0$ such that $f_1(\bar{x}) = k \cdot s$ and $f_2(\bar{x}) = -k \cdot r$. Because of the bounds on r and s, $|k| \leq (m-1)$, but this contradicts the assumed relationship between f_1 and f_2 on X. ∎

Example 4.1 (Bradley [1]): Consider the following system of constraints

$$[x_1] \, x_2^2 + 3x_3 + 2 x_2^2 x_3^2 - 15 = 0$$

$$[x_1^4] + 3 x_2 x_3^2 \qquad\qquad - 9 = 0$$

$$x_1, x_2, x_3 \geq 0 \; ; \; x_2, x_3 \text{ integer.}$$

It was shown by Bradley that the weights $r = 11$, $s = 16$ satisfy a set of his sufficient conditions for valid constraint combination, namely they are relatively prime and have the properties that $r > \max\{0, S_1\}$ where

$$S_1 \equiv \sup_x f_2(x)$$
$$\text{s.t. } \text{sgn}(s) f_1(x) \leq -|s|, \qquad x \in X$$

and $-r < \min\{0, I_2\}$, where

$$I_2 \equiv \inf_x f_2(x)$$
$$\text{s.t. } \text{sgn}(s) f_1(x) \geq |s|, \qquad x \in X.$$

Note that if, as in this case, f_1 and f_2 are bounded from below, and r and s are positive integers satisfying the above set of conditions of [1], then, for an appropriately chosen value of m, the conditions of (b) of Theorem 4.3 will hold. (For the weights $r = 11$, $s = 16$, for example, it suffices to take $m = 1$.) However, Theorem 4.3 with $m = 2$ may be applied to this example to yield the valid weights $r = 9$, $s = 14$, since the set of conditions $f_1(\bar{x}) = 14$, $f_2(\bar{x}) = -9$, $\bar{x} \in X$ is impossible, ($f_2(\bar{x}) = -9$ implies that $f_1(\bar{x}) = 3m - 15$, where m is integer) and $f_1(\bar{x}) = -14$ and $\bar{x} \in X$ imply $[\bar{x}_1] = 1$, $\bar{x}_2 = 1$, $\bar{x}_3 = 0$, for which $\bar{x}$ it is the case that $f_2(\bar{x}) \leq 6$. The weights $r = 9$, $s = 14$ do _not_ satisfy Bradley's sufficient conditions, since it is easily seen that $s = 14$ implies $S_1 = +\infty$ (the inequality constraint in the definition of S_1 does not bound x_1). ∎

Several observations regarding Theorem 4.3 are in order at this point. First note that the part of hypotheses (a) and (b) dealing with the relationship between f_1 and f_2 is also a _necessary_ condition for $E = E'(r, s)$; for, if r and s are not both 0 and there exists a $j \neq 0$ such that $f_1(\bar{x}) = -j \cdot s$ and

$f_2(\bar{x}) = j \cdot r$, then $E \neq E'(r,s)$.

For <u>non-negative</u> f_i , hypothesis (b) of Theorem 4.3 will be satisfied for $r = s = 1$, so that in this case the sum of the constraints is equivalent to the original constraints (see [3], [7], [9] for previous results and applications involving non-negative functions.)

Except for such special cases, however, the computational effort required to check hypotheses (a) or (b) depends in part on r, s and m. In particular, if r and s are small in absolute value and m is large, the hypotheses are unlikely to be satisfied, and would be difficult to check unless conditions similar to Bradley's could be easily verified.

For $m = 1$, however, the required relationship between f_1 and f_2 is vacuously satisfied, and we obtain the following Corollary, which generalizes similar results of Padberg [12] for the case in which the f_i are affine:

<u>Corollary 4.4:</u>   Let r and s be relatively prime integers such that one of the following sets of conditions is satisfied:

(a) $f_2(x) \in [-\ell_2, u_2]$ for all $x \in X$; $|r| > \max\{\ell_2, u_2\}$;

(b)   for $i = 1, 2$, there exist non-negative integers ℓ_i such that $f_i(x) \geq -\ell_i$

    for $x \in X$; $r > \ell_2$ and $s > \ell_1$;

then $E = E'(r, s)$.

Note that Corollary 4.4 generalizes Corollary 3.7 and guarantees the <u>existence</u> of weights yielding equivalent constraint combinations in the cases in which (a) <u>one</u> of the two functions is bounded both from above and below, or (b) <u>each</u> of the two functions is bounded either from above or below.

The next result corresponds to $m = 2$ and involves a condition that is generally easily checked. It also provides a generalization of the following

result of Mathews [9] for affine functions: Let $f_1(x) \equiv \sum_{i=1}^{m} \alpha_i x_i - \ell_1$, where

the α_i and ℓ_1 are <u>positive integers</u> and let $f_2(x) \equiv \sum_{i=1}^{m} \beta_i x_i - \ell_2$, where the

β_i and ℓ_2 are positive integers. If r and s are relatively prime integers

satisfying $r \geq \ell_2$ and $s \geq \ell_1$, then $\{x \mid f_1(x) = 0, f_2(x) = 0, x \geq 0$ and integer$\} =$

$\{x \mid r \cdot f_1(x) + s \cdot f_2(x) = 0, x \geq 0$ and integer$\}$.

<u>Corollary 4.5</u>: Let r and s be relatively prime positive integers and, for

$i = 1, 2,$ let $f_i(x) = g_i(x) - \ell_i$, where g_i is non-negative and integer-valued

on X and ℓ_i is a non-negative integer. If $r \geq \ell_2$, $s \geq \ell_1$, and $g_1(x) = 0$ if

and only if $g_2(x) = 0$, then $E = E'(r, s)$.

A number of other corollaries corresponding to the case $m = 2$ may be

easily derived. For example, the last statement in Corollary 4.5 could be re-

placed by the statement "If $r > \ell_2$, $s > \ell_1/2$, and $f_1(x) = -s$ implies $f_2(x) \neq r$,

then $E = E'(r, s)$." It should be kept in mind, however, that Corollaries 4.4

and 4.5 may be thought of as <u>existence</u> theorems applicable to fairly general

classes of functions, whereas the more general Theorem 4.3 identifies <u>smaller</u>

multipliers for functions obeying certain additional restrictions. (Even if the

conditions of Theorem 4.3 hold when $m = 2$ or 3, the multipliers may tend to

grow rapidly, and may reach unmanagable sizes for large systems of con-

straints. It should also be kept in mind that, from a computational viewpoint,

it is not necessarily advantageous to replace a system of constraints by a

single constraint.) The following example show that, especially for constraints

obtained from <u>inequalities</u> by adding slacks, valid constraint combination for

multipliers uniformly smaller than those of Corollary 4.4 may not be possible.

<u>Example 4.2</u> :

$$2 x_1 + 3 x_2 + x_3 \qquad\qquad = 7$$

$$x_1^2 + 4 x_2 \qquad + x_4 + x_5 = 7$$

$$X = \{x \mid x_i \geq 0 \text{ and integer, } i = 1, \ldots, 5\}.$$

Let r, s be integers in $[1,7]$, and note that by taking $\bar{x} = (0, 0, 7-s, 7, r)^T$, we obtain $f_1(\bar{x}) = -s$, $f_2(\bar{x}) = r$, where $f_1(x) = 2x_1 + 3x_2 + x_3 - 7$ and $f_2(x) = x_1^2 + 4x_2 + x_4 + x_5 - 7$. Thus, $s/r \in P'$, and Lemma 4.1 yields the inequality $E \neq E'(r, s)$. ■

If <u>no sign restrictions</u> are assumed on the coefficients in the affine case, then examples are easily constructed in the case $X = \{x \mid x \geq 0, x \text{ integer}\}$ for which P' consists of all the rationals, so that no valid combination of the constraints exists. However, on a more positive note, our next theorem shows that, even if <u>bounds</u> on the functions are <u>not</u> available, it is possible to construct valid constraint combinations if the f_i are "comparable" in an appropriate sense. (This theorem can also be considered as a generalization of the following result of Glover [4] (which, in turn, sharpens a theorem of Mathews [9]): Let $f_1(x) \equiv \sum_{j=1}^{n} \alpha_j x_j - \ell_1$, where the α_j and ℓ_1 are <u>positive</u> <u>integers</u>, and let $f_2(x) \equiv \sum_{j=1}^{n} \beta_j x_j - \ell_2$, where β_j and ℓ_2 are <u>positive</u> <u>integers.</u> If $X = \{x \mid x \geq 0 \text{ and integer}\}$, then $E = E'(1, s)$ if, for $j = 1, \ldots, n$, $s > (\ell_2 - 1)\alpha_j / \beta_j - \ell_1$ and $s > \ell_1 - (\ell_2 + 1) \alpha_j / \beta_j$.)

<u>Theorem 4.6</u>: For $i = 1, 2$, let $f_i(x) \equiv g_i(x) - \ell_i$, where g_i is integer-valued on X and ℓ_i is integer. If there exist <u>non-decreasing</u> functions w and γ such that $x \in X$ implies $f_1(x) \leq w[g_2(x)]$ and $\gamma[g_2(x)] \leq f_1(x)$, then

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$E = E'(1, s)$ for any integer $s \geq 0$ satisfying $s > w(\ell_2 - 1)$ and $s > -\gamma(\ell_2 + 1)$.

Proof: Assume the result is false, so that there exists an $\bar{x} \in X$ and an integer $k \neq 0$ such that $f_1(\bar{x}) = ks$ and $f_2(\bar{x}) = -k$. If $k \geq 1$, then $s \leq f_1(\bar{x})$ and $g_2(\bar{x}) \leq \ell_2 - 1$. Since w is non-decreasing, the latter inequality implies $w[g_2(\bar{x})] \leq w(\ell_2 - 1)$. However, this is impossible since $s \leq f_1(\bar{x}) \leq w[g_2(\bar{x})]$ and $w(\ell_2 - 1) < s$. On the other hand, if $k \leq -1$, then $f_1(\bar{x}) \leq -s$ and $\ell_2 + 1 \leq g_2(\bar{x})$. Since γ is non-decreasing, $\gamma(\ell_2 + 1) \leq \gamma[g_2(\bar{x})]$. However, this leads to a contradiction since $-s < \gamma(\ell_2 + 1)$ and $\gamma[g_2(\bar{x})] \leq f_1(\bar{x}) \leq -s$. ∎

Example 4.3: Consider the constraints:

$$x_1 - x_2 = 0$$
$$x_1 - x_2 + [sgn(x_1 - x_2)]x_1 - 2 = 0$$
$$x \in X = \{x \mid x \geq 0 \text{ and integer}\},$$

where $sgn(y) = +1$ if $y \geq 0$, and $sgn(y) = -1$ if $y < 0$. These constraints have the unique solution $x_1 = x_2 = 2$. Note that the hypotheses of Theorem 4.6 are satisfied by taking $f_1(x) = g_1(x) \equiv x_1 - x_2$, $g_2(x) \equiv g_1(x) + [sgn(g_1(x))]x_1 - 1$, $\ell_2 = 1$, $w(y) = \max\{0, y\}$ and $\gamma(y) = \min\{0, y\}$. Thus, the equality constraints may be combined with weights $r = 1$ and $s = 1 > \max\{w(0), -\gamma(2)\} = 0$ to yield tne equivalent (over X) constraint $2(x_1 - x_2) + [sgn(x_1 - x_2)]x_1 = 2$. ∎

In the case that g_1 and g_2 are <u>linear</u> and satisfy certain sign conditions, Theorem 4.6 can be applied to yield the following generalization of Glover's result:

Corollary 4.7: Let $f_1(x) \equiv \sum_{j=1}^{n} \alpha_j x_j - \ell_1$, where the α_j are <u>non-negative</u> integers, and let $f_2(x) \equiv \sum_{j=1}^{n} \beta_j x_j - \ell_2$, where the β_j are positive integers. If $X \subseteq \{x \mid x \geq 0 \text{ and integer}\}$ and $s \geq 0$ is an integer such

that $s > \bar{w} \cdot (\ell_2 - 1) - \ell_1$ (where $\bar{w} \equiv \max\{\alpha_j / \beta_j\}$) and $s > \ell_1 - \bar{\gamma} \cdot (\ell_2 + 1)$ (where

$\bar{\gamma} \equiv \min\{\alpha_j / \beta_j\}$), then $E = E'(1, s)$.

Proof: Note that if we define $g_1(x) \equiv \sum_{j=1}^{n} \alpha_j x_j$ and $g_2(x) \equiv \sum_{j=1}^{n} \beta_j x_j$, then for

$x \in X$ $\bar{\gamma} \cdot g_2(x) \leq g_1(x) \leq \bar{w} \cdot g_2(x)$, so that $\bar{\gamma} \cdot g_2(x) - \ell_1 \leq f_1(x) \leq \bar{w} \cdot g_2(x) - \ell_1$.

Thus, by defining $\gamma(y) = \bar{\gamma} \cdot y - \ell_1$ and $w(y) = \bar{w} \cdot y - \ell_1$, all of the hypotheses of

Theorem 4.6 are satisfied. ∎

More generally, if g_1 is <u>non-negative</u> on X, then the relation $\gamma[g_2(x)] \leq$

$f_1(x)$ is trivially satisfied by letting $\gamma(y) \equiv -\ell_1$, and we obtain the following

from Theorem 4.6:

Corollary 4.8: For $i = 1, 2$, let $f_i(x) \equiv g_i(x) - \ell_i$, where g_i is integer-

valued on X and ℓ_i is integer. If g_1 is non-negative on X and if there

exists a non-decreasing function w such that $x \in X$ and $g_2(x) \leq \ell_2 - 1$ imply

$f_1(x) \leq w[g_2(x)]$, then $E = E'(1, s)$ for any integer $s > 0$ satisfying $s > \ell_1$, $s >$

$w(\ell_2 - 1)$.

Proof: Note from the proof of Theorem 4.6 that the inequality $f_1(x) \leq w[g_2(x)]$

is required only in the case $g_2(x) \leq \ell_2 - 1$. With this observation the Corollary

results from taking $\gamma(y) \equiv -\ell_1$. ∎

Example 4.4: Consider the following system of constraints (obtained from Bradley's

example by changing one term and interchanging the order of the resulting equa-

tions):

$$[x_1^4] + 3x_2 x_3^2 \qquad - 9 = 0$$

$$[x_1^5] + 3x_3 + 2x_2^2 x_3^2 - 15 = 0 .$$

It may be verified that, in applying Theorem 4.8 to this example, it is possible

to use $w(y) = y - 9$, so that a valid set of multipliers is $r = 1$, $s = 10$.

It might be noted that the proof of Theorem 4.6 also goes through if w and γ are assumed to be <u>increasing</u> and the conditions on s are replaced by the set of conditions $s \geq 0$, $s \geq w(\ell_2)$, and $s \geq -\gamma(\ell_2)$. These conditions have the interesting property that if either of $w(\ell_2)$ or $-\gamma(\ell_2)$ is negative, then it is possible to conclude that E is <u>empty</u>. For, if $\hat{x} \in E$, then $g_2(\hat{x}) = \ell_2$ and $f_1(\hat{x}) = 0$ so that $0 = f_1(\hat{x}) \leq w(\ell_2)$ and $\gamma(\ell_2) \leq f_1(x) = 0$.

In the linear case, an extension of Corollary 4.7 can be obtained for the case in which <u>both</u> sets of coefficients α_j and β_j are merely assumed to be <u>non-negative.</u> Note that if for some k , $\alpha_k = \beta_k = 0$, then the variable x_k is unrestricted both before and after any constraint combination, and thus has no effect on the relation between E and $E'(r, s)$.

<u>Corollary 4.9</u>: Let $f_1(x)$ and $f_2(x)$ be defined as in Corollary 4.7, let α_j and β_j be <u>non-negative</u> integers, and let J be the set of j satisfying $\alpha_j + \beta_j > 0$. If $X \subseteq \{x \mid x \geq 0$ and integer$\}$ and $\hat{s} \geq 0$ is an integer such that $\hat{s} > \hat{w} \cdot (\ell_1 + \ell_2 - 1) - \ell_1$ (where $\hat{w} \equiv \max_{j \in J} \{\alpha_j / (\alpha_j + \beta_j)\}$) and $\hat{s} > \ell_1 - \hat{\gamma}(\ell_1 + \ell_2 + 1)$ (where $\hat{\gamma} \equiv \min_{j \in J} \{\alpha_j / (\alpha_j + \beta_j)\}$), then $E = E'(1 + \hat{s}, \hat{s})$.

<u>Proof:</u> The constraints $f_1(x) = 0$ and $f_2(x) = 0$ are equivalent to the constraints $f_1(x) = 0$ and $f_1(x) + f_2(x) = 0$. The result then follows from the application of Corollary 4.7 to the latter system, ignoring any 0 columns for the reasons cited above. ∎

For the final result of this section, note that if there exist y_1 and y_2 such that $y_1 \alpha_j + y_2 \beta_j > 0$ for all j , then the original system can be transformed into an equivalent system in which all coefficients are positive, so that Corollary 4.7 may be applied. Moreover, the existence and values of suitable y_1 and y_2 may be determined via linear programming by solving the system of inequalities

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$y_1 \alpha_j + y_2 \beta_j \geq 1$ $(j = 1, \ldots, n)$ (to avoid trivial cases, assume $\alpha_j^2 + \beta_j^2 > 0$).

<u>Theorem 4.10:</u> Let $f_1(x) \equiv \sum_{j=1}^{n} \alpha_j x_j - \ell_1$ and $f_2(x) \equiv \sum_{j=1}^{n} \beta_j x_j - \ell_2$. If $X \subseteq$ $\{x \mid x \geq 0 \text{ and integer}\}$ and there exist y_1 and y_2 such that $y_1 \alpha_j + y_2 \beta_j \geq 1$ for all j, then there exist r and s such that $E = E'(r, s)$.

Of course, the existence of y_1 and y_2 satisfying $y_1 \alpha_j + y_2 \beta_j > 0$ for all j implies finiteness of E. In the next section it will be shown that if $X = \{x \mid x \geq 0 \text{ and integer}\}$, then finiteness of E is also a <u>necessary</u> condition for the existence of r and s such that $E = E'(r, s)$, providing a partial converse to Theorem 3.3.

5. Finiteness as a Necessary Condition.

Sufficient conditions (including finiteness of the set X) for equivalent formulations of various types were established in the preceding sections. In this section we will show that for <u>linear</u> equations over $Z_+^n \equiv \{x \mid x \in \mathbb{R}_+^n, x \text{ integer}\}$, finiteness of the feasible set is a <u>necessary</u> condition (except in certain trivial cases) for the existence of equivalent constraint combinations.

The result will first be obtained in the case of two equations and then generalized to systems of m equations. The following Lemma, which may be thought of as a "non-standard" theorem of the alternative is needed for the two equation case (geometrically, the alternatives correspond to the cases in which the set $\{z \mid z = Ax \text{ for some } x \geq 0\}$ is and is not a "pointed" cone):

<u>Lemma 5.1:</u> Let A be a $2 \times n$ matrix with linearly independent rows and no 0 columns. Then exactly one of the following alternatives must hold:

(i) there exists a vector y such that $y^T A > 0$:

(ii) for each $b \in \mathbb{R}^2$, there exists an $x \geq 0$ such that either $Ax = b$ or $Ax = -b$.

Proof: Both (i) and (ii) cannot hold simultaneously, for let b be chosen so that $b \neq 0$ and $by = 0$. Then if $x \geq 0$ is such that $Ax = b$ or $Ax = -b$, then $0 < (y^T A)x = y^T(Ax) = 0$, a contradiction. Suppose that (ii) does not hold, and let b be chosen so that neither the system $Ax = b$, $x \geq 0$ nor the system $Ax = -b$, $x \geq 0$ is solvable. By the Farkas Lemma, there exists vectors u and v such that $u^T A \geq 0$, $u^T b < 0$ and $v^T A \geq 0$, $v^T b > 0$. We will show that $(u+v)^T A > 0$. First note that since the rank of A is 2, A contains 2 linearly independent columns, so that $u^T A \neq 0$ and $v^T A \neq 0$. Thus, $u \neq \rho \cdot v$ for any $\rho \leq 0$. Moreover, $u^T b < 0$ and $v^T b > 0$ imply $u \neq \rho \cdot v$ for any $\rho > 0$. Thus, if q is a column of A such that $u^T q = 0$, then $v^T q > 0$. (Here we take advantage of a property of $\mathbb{R}^2$ - this result does not extend to the case in which A has rank 2 and is $m \times n$ for $m > 2$). The result follows by taking $y = (u + v)$. ∎

Theorem 5.2. Let $E \equiv \{x \mid Ax - \ell = 0, x \in X\}$, where A is a $2 \times n$ integer matrix with linearly independent rows and no 0 column, ℓ is an integer vector, and $X = \{x \mid x \geq 0, x \text{ integer}\}$. If $E \neq \phi$, then the following are equivalent:

(a) there exist integers r, s such that $E = E'(r, s)$;

(b) there exists a vector y such that $y^T A > 0$;

(c) E is finite.

Proof: The relation (b) $\Rightarrow$ (a) follows from Theorem 4.10.

To prove that (a) $\Rightarrow$ (b), let $b = (-s, r)^T$ and note that since $A \neq 0$, $E \neq \mathbb{R}^n$ and thus r and s cannot both be 0, so $b \neq 0$. We will now show that neither of the systems $Ax = b$, $x \geq 0$ and $Ax = -b$, $x \geq 0$ has a solution. For, if $Ax = b$, $x \geq 0$ had a solution, it would have a (rational) basic feasible

solution x^*, so for an appropriate positive integer k, kx^* would be integer and $Ax^* = kb$. Letting $x' \epsilon E$ then, $A(x' + x^*) = kb + \ell$, where $\ell = (\ell_1, \ell_2)^T$, so that $(x' + x^*)$ is in $E'(r,s)$ but not in E. A similar argument applies if $Ax = -b$, $x \geq 0$ is assumed solvable. The conclusion (a) $\Longrightarrow$ (b) then follows from Lemma 5.1.

Clearly (b) $\Longrightarrow$ (c), so we need only show that (c) $\Longrightarrow$ (b). If (b) did not hold, then by Gordan's theorem, there would exist a non-zero solution of the system $Ax = 0$, $x \geq 0$, and hence a rational solution, which in turn can be used to contradict the finiteness of E. $\blacksquare$

It should be noted that if the matrix A does not have linearly independent rows, then there also exist weights such that $E = E'(r,s)$. This is easily seen by considering separately the cases (1) rank $A = 0$, (let $r = \ell_1$, $s = \ell_2$) (2) rank $A = 1$, $E = \phi$ (choose r and s such that $(r,s) A = 0$ and $(r,s)\ell \neq 0$), and (3) rank $A = 1$, $E \neq \phi$ (choose r and s such that $(r,s) A \neq 0$). Moreover, the hypothesis that A contains no 0 columns merely rules out trivial cases in which variables do not appear in the equations. The presence of such columns has no effect on the validity of constraint combinations, so that from a computational viewpoint one need only consider the non-zero columns.

In extending Theorem 5.2 to systems of equations, the following direct consequence of the theorem is useful.

Corollary 5.3: Let the hypotheses of Theorem 5.2 hold, and assume in addition that E is infinite; then, for every pair r,s of integers, E is a proper subset of $E'(r,s)$.

Corollary 5.4: Let $E^* \equiv \{x | A^* x - \ell^* = 0, x \epsilon X\}$, where A^* is an $m \times n$ integer matrix with rank at least 2 and no 0 columns, ℓ^* is an integer vector, and $X = \{x | x \geq 0, x \text{ integer}\}$. If $E^* \neq \phi$, then the following are equivalent:

(a) there exists an integer vector q such that

$$E^* = \{x \mid q^T A^* x - q^T \ell^* = 0, \ x \in X\} \ ;$$

(b) there exists a vector y such that $y^T A^* > 0$;

(c) E^* is finite.

Proof: The equivalence of (b) and (c) follow by arguments analogous to those of the proof of Theorem 5.2, as does (b) $\Longrightarrow$ (a), so we need only show that (a) $\Longrightarrow$ (c). Suppose that (a) holds, but that (c) is false. Without loss of generality, let us assume that the rows of A^* have been ordered so that the first two rows of A^* are linearly independent and let A be the matrix consisting of $a_1 = a_1^*$, the first row of A^* and $a_2 = \sum_{i=2}^{m} q_i a_i^*$, where a_i^* is the ith row of A^*. Similarly, let ℓ be the 2-vector whose first component is the first component of ℓ^* and whose second component is $\sum_{i=2}^{m} q_i \ell_i^*$. Note that (a) implies $E^* = E'(q_1, 1) = E$ (here the notation of Theorem 5.2 is being used). Since E^* is infinite, we can obtain a contradiction via Corollary 5.3 if A has rank 2 and no 0 columns. Note that A can have no 0 columns, for, if column j of A were 0 and $x' \in E^*$, then for any positive integer k, the vector $x' + k \cdot e_j$, where j is the jth unit vector, would be in E but not in E^* (since the jth column of A^* is non-zero), contradicting $E = E^*$. If the rows of A were linearly dependent, then since $E^* \neq \varphi$, it would be the case that $E^* = \{x \mid a_1^* x - \ell_1^* = 0, \ x \in X\}$. However, by Corollary 5.3 with $r = 1$ and $s = 0$, the set on the RHS properly contains the set $\{x \mid a_1^* x - \ell_1^* = 0, \ a_2^* x - \ell_2^* = 0, \ x \in X\}$, which, in turn, contains E^*, yielding a contradiction. $\blacksquare$

If $X \subseteq Z_+^n$ and there exists a y such that $y A > 0$, then by Theorem 4.10 there exist weights such that $E = E'(r, s)$. If such a y does not exist, then the cone $K \equiv \{z \mid z = Ax, x \geq 0\}$ consists either of the origin (if $A = 0$), a line (if A has rank 1), a half-space, or all of $\mathbb{R}^n$. (Note that the latter case will hold if and only if neither

the system $yA > 0$ nor the system $yA \leq 0$ has a non-zero solution.) The

following theorem gives sufficient conditions for $E = E'(r, s)$ in the case $K = \mathbb{R}^n$:

Theorem 5.6: Let $E = \varphi$ and $X = Z_+^n$. If $\{z | Ax = z, x \geq 0\} = \mathbb{R}^n$ and there exists

an integer vector $\bar{y} = (\bar{y}_1, \bar{y}_2)$ with $\bar{y}_1$ and $\bar{y}_2$ relatively prime such that the

system $Ax = \bar{y}$ has a solution in Z_+^n, then $E = E'(\bar{y}_2, -\bar{y}_1)$.

Proof: Suppose that $E'(\bar{y}_2, -\bar{y}_1) \neq \varphi$, so that there exists an $\bar{x} \in Z_+^n$ such that

$A\bar{x} - \ell = k\bar{y}$ for some non-zero integer k. Let $x' \in Z_+^n$ satisfy $Ax' = \bar{y}$, and choose

an integer $\rho > 0$ such that $Ax = (-\rho)\bar{y}$ has a solution x'' in Z_+^n. If $k > 0$, choose

an integer q such that $q \cdot \rho \geq k$, and obtain a contradiction from the relations

$$A(\bar{x} + (q \cdot \rho - k)x' + qx'') - \ell = k\bar{y} + (q \cdot \rho - k)\bar{y} - (q \cdot \rho)\bar{y} = 0 \ .$$

If $k < 0$, a contradiction is similarly obtained by noting that $A(\bar{x} + kx') - \ell = 0$. ∎

Corollary 5.7: If $E = \varphi$, $\{z | Ax = z, x \geq 0\} = \mathbb{R}^n$, and A contains a column with

relatively prime elements α_j, β_j, then $E = E'(\beta_j, -\alpha_j)$.

Given a method for determining feasibility or infeasibility in the single

constraint (knapsack) case, one may use an extension of Theorem 5.6 to determine

whether or not $E = \varphi$. Once all hypotheses (other than $E = \varphi$) of Theorem 5.6 have

been verified, then the determination that $E'(\bar{y}_2, -\bar{y}_1) = \varphi$ implies $E = \varphi$ (since

$E \subseteq E'(\bar{y}_2, -\bar{y}_1)$), and the determination that $E'(\bar{y}_2, -\bar{y}_1) \neq \varphi$ implies that E is infinite.

5. Summary.

We have shown how, under various discreteness hypotheses equivalent

yet "simpler" formulations may be obtained for systems of constraints. These

results generalize and extend related constraint transformation results of Gould

and Rubin, Hammer and Rudeanu, Bradley, Glover, and others, and also yield

some interesting insights into stability (an instability) properties of integer programs.

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Two types of "simplifications" are considered for constraints over discrete sets: (1) replacing real data by "equivalent" rational data, and (2) collapsing a system of linear or nonlinear equations into an "equivalent" single equation. Such transformations are not only of computational interest, but also provide some interesting insights into stability properties of integer programs.

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